20/4/23 MATH2060 lecture

Announcements:

- HW due April 28 11am
- FINAL: Thursday 11 May

Taylor's Series

Recall: By repeated application of differentiation Thm for power series,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}, \quad x \in (-R,R), \quad \forall k \in \mathbb{N}.$$

Thm 9.4.13 (Uniqueness Thm): If $\sum a_n x^n$, $\sum b_n x^n$ converge to the same function f on $(-r,r)$, $r>0$, then $a_n = b_n$ $\forall n \in \mathbb{N}$.

Pf: By remark above, $f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$

$$f^{(k)}(0) = \frac{k!}{(k-k)!} a_k \Rightarrow a_k = \frac{1}{k!} f^{(k)}(0).$$

Similarly, $b_k = \frac{1}{k!} f^{(k)}(0)$. //

Taylor Series:

Let f have derivatives of all orders at a point $c \in \mathbb{R}$. Then by above, we can have the power series $g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$.

which will satisfy $g^{(n)}(c) = f^{(n)}(c)$. $\forall n \in \mathbb{N}$.

A few issues: a priori

1) We don't know whether the power series converges (except at c).

2) If it converges, we don't know if it agrees with f on $(-R, R)$.

(Exercise 9.4.12)

Def: We say $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is the Taylor Series Expansion of f at c if $\exists R > 0$ s.t.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \text{ converges to } f(x) \text{ on } (c-R, c+R).$$

$\frac{f^{(n)}(c)}{n!}$ are called Taylor Coefficients.

Rmk: 1) i.e. need in Taylor's Thm (Thm 6.4.1), $R_n(x) \to 0$ on $(c-R, c+R)$

2) By Uniqueness Thm, if Taylor Series Expansion exists, then it is unique.

Examples 9.4.4

a) $f(x) = \sin x$, $x \in \mathbb{R}$.

$$f^{(n)}(x) = \begin{cases} (-1)^k \sin x & \text{if } n = 2k \\ (-1)^k \cos x & \text{if } n = 2k+1 \end{cases}$$

In particular, at $c=0$, $f^{(n)}(0) = \begin{cases} 0 & \text{if } n=2k \\ (-1)^k & \text{if } n=2k+1 \end{cases}$

Furthermore, by Taylor's Thm: $R_n(x)$ satisfies

$$|R_n(x)| \leq \frac{|f^{(n+1)}(c_1)|\,|x|^{n+1}}{(n+1)!} \quad \text{for some } c_1 \text{ between } x \text{ and } 0.$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n\to\infty. \quad \forall x\in\mathbb{R}.$$

Taylor Expansion of $f(x) = \sin x$ at 0, is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x\in\mathbb{R}.$$

Differentiation Thm $\Rightarrow$ $\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\forall x\in\mathbb{R}$

In this example, we used $R_n(x)$ from Taylor's Thm to determine the radius of convergence.

If we want to use $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ isn't well-defined.

For $\rho = \limsup\left(|a_n|^{\frac{1}{n}}\right) = \lim_{k\to\infty}\left(\frac{1}{(2k+1)!}\right)^{\frac{1}{2k+1}} = 0$. So $R = +\infty$.

2) $g(x) = e^x$, $x \in \mathbb{R}$. Then $g^{(n)}(x) = e^x$ $\forall n \in \mathbb{N}$. $g^{(n)}(0) = 1$

$\Rightarrow$ Coefficients are $\frac{1}{n!}$

$$|R_n(x)| \leq \frac{e^c}{(n+1)!}|x|^{n+1} \quad \text{for some } c \text{ between } x \text{ and } 0.$$

$$\leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n\to\infty, \quad \forall x \in \mathbb{R}$$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\forall x \in \mathbb{R}$. is the Taylor expansion of e^x at 0.

Furthermore, $e^x = e^c e^{x-c} = e^c \sum_{n=0}^{\infty} \frac{1}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{e^c (x-c)^n}{n!}$, $\forall x \in \mathbb{R}$

is the Taylor Expansion for e^x centred at $x=c$.

Rmk: 1) This argument implies the radius of convergence is $+\infty$

$$\rho = \limsup\left(|a_n|^{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0 \Rightarrow R = +\infty.$$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} n+1 = +\infty.$$